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In-phase and out-of-phase break-up of two immersed liquid threads under influence of surface tension

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Abstract

The dynamical behaviour of two infinitely long adjacent parallel liquid threads immersed in a fluid is considered under influence of small initial perturbations. Assuming all fluids to behave Newtonian, we used the creeping flow approximation, which resulted in Stokes equations. Applying cylindrical coordinates and separation of variables, and writing the dependence on the azimuthal direction in the form of a Fourier expansion, we obtained general representations of the equations for both the threads and the surrounding fluid. Substitution of these expressions into the boundary conditions leads to an infinite set of linear equations for the unknown coefficients. Its solutions for the lowest two orders of the Fourier expansion, the so-called zero- and first-order solutions, are presented. Much attention is paid to the (in)stability of the configuration, in terms of the so-called growth rate of the disturbance amplitudes. The growth rate of these amplitudes determines the behaviour of the break-up process of the threads. It turns out that this breaking up occurs either in-phase or out-of-phase. This depends on the viscosity ratio of the fluids and on the distance between the threads. These findings agree with experimental observations. The results of the present work also show that the zero-order solution yields the qualitatively correct insight in the break-up process. The extension to a one order higher expansion only leads to relatively small quantitative corrections. © 2002 Éditions scientifiques et médicales Elsevier SAS. All rights reserved.

1. Introduction

The demand for new synthetic materials increases and becomes more specific, nowadays. New synthetic materials may be produced by blending different types of polymers. The material properties of a polymer blend are strongly related to its morphology determined by the blending process in the extruder. Therefore, the eventual material properties can be predicted only if a thorough understanding of this blending process is available.

In the blending process under consideration, relatively big drops of one material are immersed into a shear flow of a second material. Due to dominant shear stresses, long threads are formed. At some moment a thread may become so thin and its radius so small that the inter-facial stress becomes important. The effect of the latter is a tendency to attain the drop-shape. Initiated by perturbations, waves may develop along the thread. Driven by surface tension, these waves may grow in amplitude or attenuate, depending on the stability of the system. In an unstable state, the thread will eventually break up into an array of small spherical droplets. This break-up of a cylindrical thread has interested many scientists.

The problem of breaking up of one isolated liquid thread in a fluid, for the case that both thread and surrounding fluid behave Newtonian, has already been solved long ago. If the fluid is initially at rest, the problem of stability is purely geometrical; that is, instability means that the surface area can be reduced by an infinitesimal surface deformation. The classical example is due to Plateau [1,2] who studied experimentally the instability of an infinitely long cylindrical fluid thread (or: filament). Plateau

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found that the system is unstable if the wavelength of a perturbation is greater than π times the diameter of the jet. From this it may seem as if disturbances of very long wavelength are the most rapidly growing ones, as they reduce the surface area most. But this leaves out the dynamics of the problem. In fact, the surface tension has to overcome both inertia and viscous dissipation.

The dynamical description of the problem, in terms of linear stability theory, was first introduced by Rayleigh [3,4]. He considered the stability of a long cylindrical column of an incompressible perfect fluid under the action of capillary forces, neglecting the effect of the surrounding fluid. His result is in accordance with the previous result of Plateau. In his paper, Rayleigh developed the important concept of the mode of maximum instability. The degree of instability, which characterizes the growth rate of the disturbance amplitude and as indicated by the value of q in the exponential e^{qt} to which the motion is assumed to be proportional, depends upon the value of wavelength of the disturbances λ . For the inertial case, so with viscosity ignored, it reaches a maximum when λ equals 4.51 times the diameter of the cylinder. Tomotika [5,6], generalized Rayleigh's analysis to include viscosity for both the fluid column and the surrounding fluid. He derived a general low velocity equation relating the growth rate of a disturbance to liquid properties, thread diameter, and wavelength of the disturbance by writing the equations of motions for an incompressible Newtonian fluid for both phases and relating the two boundary conditions at the interface. He found that if the ratio of viscosities of the two fluids is neither zero nor infinity the maximum instability always occurs at a certain definite value of the wavelength of the assumed initial perturbation. Moreover, he concluded that the formation of droplets of definite size is to be expected. Tomotika's analysis has been developed further by, for example, Meister et al. [7], Kinoshita et al. [8], and Stone et al. [9]. A wide-ranging review of the dynamics of the break-up process of one thread is given by Eggers [10]. The effect of viscosity and surface tension on the stability of the plane interface between two fluids has also been studied by Mikaelian [11,12]. Using a moment equation approach, he found that perturbations at the interface show damped oscillations when viscosity and surface tension are present. A model describing the nonlinear dynamics of one thread when surface tension drives the motion is discussed by Papageorgiou [13]. Using nonlinear long-wave theory, he is able to describe phenomena such as jet pinching, which are beyond the scope of linear theory. The pinching effect, the scaling behaviour of singularities in nonlinear systems, and drop break-up are studied to some extent in [14–20]. In the present paper we restrict ourselves to linear stability theory for a set of two threads, since the literature is still lacking for such systems.

Most blends contain a large volume fraction of the dispersed phase (threads). In the experimental practice, it turns out that the interactions between the threads are of essential importance for the way they break up. In experiments reported in Knops [21] and Knops et al. [22] it is observed that neighbouring threads may break up in-phase or out-of-phase. Both possibilities are shown in Figs. 1 and 2. Knops et al. focus their attention on the phase difference α between two neighbouring threads of equal radius a and the distance b . Using the principle of minimal dissipation of energy, they found that the dissipation term due to the interaction of the threads depends on the relative distance $\hat{b} = b/a$ between the threads. Taking the ratio of viscosities equal to 0.91, they found a critical distance $\hat{b} = 3.4$. For $\hat{b} < 3.4$, the dissipation is minimal for $\alpha = \pi$, which implies that the threads will break up out-of-phase. For $\hat{b} > 3.4$, the dissipation is minimal for $\alpha = 0$ implying the threads will break up in-phase.



Fig. 1. Experimental break-up behaviour (from [21,22]). (a) Out-of-phase break-up of PA-6 (polyamide) threads in a matrix of PS-N7000 (polystyrene); (b) in-phase break-up of PA-6 (polyamide) threads in a matrix of PS-N1000 (polystyrene).

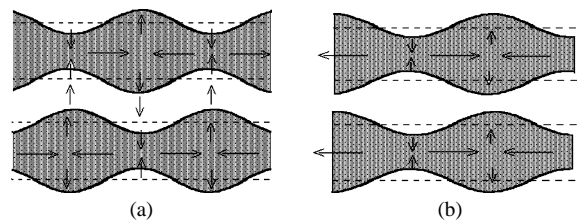


Fig. 2. A sketch of two typical examples of break-up behaviour observed in the experiment from [21,22]. (a) Out-of-phase; (b) in-phase.

In this paper, we study the origin of the phenomena described above and the dependence on the geometrical and rheological parameters. To that end, the dynamical behaviour of two adjacent parallel threads immersed in a polymeric fluid (the matrix) under the action of both surface tension and viscous forces is considered. The threads are initially cylindrical, but due to thermal fluctuations disturbances are always present. The fluids are assumed to be at rest except for small disturbances which are assumed to develop slowly. This implies that the velocities and the shear rates will be small. Under these conditions, it is justified to model both the threads and the matrix as Newtonian fluids and to use the creeping flow approximation, resulting in Stokes equations. These equations are solved by means of separation of variables in two systems of cylindrical coordinates, each one connected to one of the threads. In this, the dependence on the azimuthal directions is written in the form of Fourier expansion. Substitution of the general solution into the boundary conditions yields an infinite set of linear equations for the unknown coefficients. This set is solved using the method of moments.

Based on the solution found above, the instability (the growth rate) of the initial disturbances is investigated. If the initial disturbances increase in time, the initial state is unstable resulting in breaking up. In the present work, the instability of the threads is examined based on both a zero-order and a first-order Fourier expansion. The zero-order analysis follows the lines presented in [21], but avoids some unreliable assumptions. This refinement leads to expressions that are simpler to evaluate. The model confirms that the break up can indeed occur in-phase as well as out-of-phase. This depends on the viscosity ratio of the two fluids and on the distance between the threads. All this is in correspondence with experimental observations. In an analogous way the first-order expansion is evaluated. The results show that the zero-order solution is already quite reliable to describe the essential features.

The paper is organized as follows. In Section 2, the mathematical model is derived, the general solution written in terms of Fourier expansions and a general scheme for solving the unknown coefficients is presented. In this scheme an infinite set of equations is reduced to a finite set of equations. In Section 3, the zero- and first-order solutions are derived. As an illustration, the zero-order solution is worked out into detail and the explicit solution for the unknown coefficients is presented. The core of the paper is Section 4. Based on the solution given in Section 3, the instability of the threads is investigated. Conclusions are given in the last section.

2. Mathematical model and solution methodology

Consider two infinitely long parallel threads, both with viscosity μ_d , which are surrounded by a viscous fluid with viscosity μ_c , where the indices c and d refer to the continuous phase (surrounding fluid) and the disperse phase (threads), respectively. We denote the ratio of the viscosities by $\mu = \mu_d/\mu_c$. Let b be the distance between the two centers of the threads. In Fig. 3(a) radial cross-section is sketched. The indices 1 and 2 refer to thread 1 and thread 2, respectively. Two cylindrical coordinate systems will be used, (r_1, ϕ_1, z_1) centered along the axis of thread 1, (r_2, ϕ_2, z_2) centered along the axis of thread 2.

If the perturbations are still small, the threads are close to cylinders. The disturbed radii are written as a sum of modes, that are periodic in z with wave number k . Inspired by the experimental observations (see Fig. 1), we consider for the moment only the special sub-set of disturbances for which both wave numbers are the same, $k_1 = k_2 = k$. Because the problem is not axisymmetric, the disturbances will also depend on ϕ . This dependence is written as a Fourier expansion. Thus, we consider two disturbed threads with radii R_1 and R_2 given by

$$\begin{aligned} R_1(\phi_1, z, t) &= a + \left(\sum_{n=0}^{\infty} \varepsilon_{1n}(t) \cos n\phi_1 \right) \cos kz, \\ R_2(\phi_2, z, t) &= a + \left(\sum_{n=0}^{\infty} \varepsilon_{2n}(t) \cos n\phi_2 \right) \cos(kz - \alpha). \end{aligned} \quad (1)$$

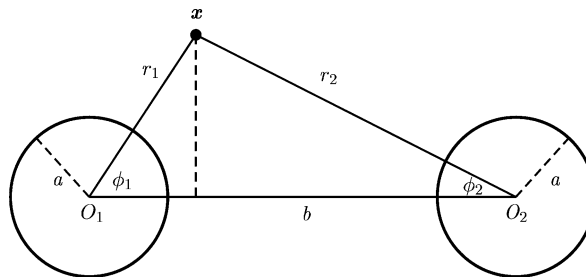


Fig. 3. A radial cross-section of the two threads and the corresponding two cylindrical coordinate systems.

Since z_1 and z_2 are identical, we dropped the index in the z -coordinates. Here, a is the mean radius of the threads, ε_{1n} and ε_{2n} are the time dependent amplitudes of the modes, and α is a phase difference between the modes at thread 1 and the modes at thread 2. The parameter α is unknown in advance.

The polymer fluids are assumed to behave Newtonian. This is a good approximation for slow deformation rates. Since the dynamics is only driven by surface tension, the velocities and the shear rates will be small. Also the length scales are very small. The Reynolds number is therefore small and the creeping flow approximation may be used. For this slow and incompressible flow, the system is described by Stokes equations:

$$\nabla \cdot \mathbf{v} = 0, \quad \nabla p = \mu \Delta \mathbf{v}, \quad (2)$$

where p is the pressure, μ is the viscosity, and $\mathbf{v} = (u, v, w)$, with u, v and w the velocity components in radial, azimuthal and axial directions, respectively. Assuming that separation of variables is applicable and expressing the dependence in ϕ in terms of Fourier modes, we propose as general expressions for the solution inside thread 1:

$$\begin{aligned} p_1(r_1, \phi_1, z, t) &= \sum_{n=0}^{\infty} p_{1n}(r_1, t) \cos n\phi_1 \cos kz, & u_1(r_1, \phi_1, z, t) &= \sum_{n=0}^{\infty} u_{1n}(r_1, t) \cos n\phi_1 \cos kz, \\ v_1(r_1, \phi_1, z, t) &= \sum_{n=1}^{\infty} v_{1n}(r_1, t) \sin n\phi_1 \cos kz, & w_1(r_1, \phi_1, z, t) &= \sum_{n=0}^{\infty} w_{1n}(r_1, t) \cos n\phi_1 \sin kz. \end{aligned} \quad (3)$$

The analogue of (3) for thread 2 is obtained by replacing 1 by 2 and shifting z over α . For example, the pressure inside thread 2 is given by

$$p_2(r_2, \phi_2, z, t) = \sum_{n=0}^{\infty} p_{2n}(r_2, t) \cos n\phi_2 \cos(kz - \alpha). \quad (4)$$

For the continuous phase, we should note that the local state is influenced by the presence of both threads. As an ansatz, we represent the solution outside both threads by the sum of an expansion like the one in (3) in terms of cylindrical coordinates centered at thread 1 and a similar expansion in terms of cylindrical coordinates centered at thread 2. The addition of the velocities from both expansions is done vectorially. Then, we obtain

$$\begin{aligned} p_c &= p_{c1} + p_{c2}, \\ u_c &= u_{c1} - u_{c2} \cos(\phi_1 + \phi_2) + v_{c2} \sin(\phi_1 + \phi_2), \\ v_c &= v_{c1} + u_{c2} \sin(\phi_1 + \phi_2) + v_{c2} \cos(\phi_1 + \phi_2), \\ w_c &= w_{c1} + w_{c2}. \end{aligned} \quad (5)$$

Note that here u_c and v_c are the radial and azimuthal components with respect to the coordinate system of thread 1. The indices cj ($j = 1, 2$) denote the expansion in terms of cylindrical coordinates centered at thread j . For example, the radial velocity of the continuous phase due to thread 1 is written as

$$u_{c1} = \sum_{n=0}^{\infty} u_{c1n}(r_1, t) \cos n\phi_1 \cos kz. \quad (6)$$

Substituting the expansions (3) into Stokes equations (2), we obtain the solutions for the n th mode in terms of the modified Bessel functions of first and second kind, I_n and K_n , respectively, of order n . Application of the condition of boundedness of the solutions both in the origin and at infinity leads to the insight that inside thread j ($j = 1, 2$)

$$\begin{aligned} p_{jn}(r_j, t) &= 2\mu_d A_{jn}(t) I_n(kr_j), \\ u_{j0}(r_j, t) &= A_{j0}(t) r_j I_0(kr_j) - \left[B_{j0}(t) + \frac{2}{k} A_{j0}(t) \right] I_1(kr_j), \\ u_{jn}(r_j, t) &= A_{jn}(t) r_j I_n(kr_j) - \left[B_{jn}(t) + \frac{1}{k} (n+2) A_{jn}(t) \right] I_{n+1}(kr_j) + \frac{C_{jn}(t)}{r_j} I_n(kr_j) \quad (n \geq 1), \\ v_{jn}(r_j, t) &= - \left[B_{jn}(t) + \frac{1}{k} (n+2) A_{jn}(t) + \frac{k}{n} C_{jn}(t) \right] I_{n+1}(kr_j) - \frac{1}{r_j} C_{jn}(t) I_n(kr_j), \\ w_{jn}(r_j, t) &= -A_{jn}(t) r_j I_{n+1}(kr_j) + B_{jn}(t) I_n(kr_j). \end{aligned} \quad (7)$$

For the continuous phase, we have

$$\begin{aligned}
 p_{cjn}(r_j, t) &= 2\mu_c D_{jn}(t) K_n(kr_j), \\
 u_{cj0}(r_j, t) &= D_{j0}(t) r_j K_0(kr_j) + \left[E_{j0}(t) + \frac{2}{k} D_{j0}(t) \right] K_1(kr_j), \\
 u_{cjn}(r_j, t) &= D_{jn}(t) r_j K_n(kr_j) + \left[E_{jn}(t) + \frac{1}{k} (n+2) D_{jn}(t) \right] K_{n+1}(kr_j) + \frac{F_{jn}(t)}{r_j} K_n(kr_j) \quad (n \geq 1), \\
 v_{cjn}(r_j, t) &= \left[E_{jn}(t) + \frac{1}{k} (n+2) D_{jn}(t) + \frac{k}{n} F_{jn}(t) \right] K_{n+1}(kr_j) - \frac{1}{r_j} F_{jn}(t) K_n(kr_j), \\
 w_{cjn}(r_j, t) &= D_{jn}(t) r_j K_{n+1}(kr_j) + E_{jn}(t) K_n(kr_j).
 \end{aligned} \tag{8}$$

To find the values for the unknown coefficients A_{jn} , B_{jn} , D_{jn} and E_{jn} , (7) and (8) must be substituted into the boundary conditions at the interface. These boundary conditions are given by

- no slip at the interfaces: $\llbracket u \rrbracket = \llbracket v \rrbracket = \llbracket w \rrbracket = 0$; with $\llbracket u \rrbracket$ the jump in u , etc.;
- continuity of the tangential stresses at the interfaces: $\llbracket \sigma_{\phi r} \rrbracket = \llbracket \sigma_{zr} \rrbracket = 0$;
- discontinuity of the normal stresses at the interfaces due to the interfacial tension: $\llbracket \sigma_{rr} \rrbracket = \Delta \sigma_{rr,j}$.

These conditions are evaluated up to linear order, which implies that all evaluations are taken with respect to $r = a$, the initial radius of the thread. Moreover, $\Delta \sigma_{rr,j}$ is the interfacial tension at thread j , defined as

$$\Delta \sigma_{rr,j} = \sigma(\kappa_1 + \kappa_2), \tag{10}$$

where σ is the surface tension and κ_1 and κ_2 are the two main curvatures which in linear approximation are given by

$$\begin{aligned}
 \kappa_1 &= \frac{\partial^2 R_j / \partial z^2}{(1 + (\partial R_j / \partial z)^2)^{3/2}} \approx \frac{\partial^2 R_j}{\partial z^2}, \\
 \kappa_2 &= -\frac{R_j^2 + 2(\partial R_j / \partial \phi)^2 - R_j(\partial^2 R_j / \partial \phi^2)}{(R_j^2 + (\partial R_j / \partial \phi)^2)^{3/2}} \approx \frac{1}{R_j^2} \frac{\partial^2 R_j}{\partial \phi^2} - \frac{1}{R_j}.
 \end{aligned} \tag{11}$$

Hence, combining (1) and (11) we find for the linearized perturbed interfacial tensions

$$\begin{aligned}
 \Delta \sigma_{rr,1}(\phi_1, z, t) &= \sigma \left(\sum_{n=0}^{\infty} \frac{1 - (ak)^2 - n^2}{a^2} \varepsilon_{1n}(t) \cos n\phi_1 \right) \cos kz, \\
 \Delta \sigma_{rr,2}(\phi_2, z, t) &= \sigma \left(\sum_{n=0}^{\infty} \frac{1 - (ak)^2 - n^2}{a^2} \varepsilon_{2n}(t) \cos n\phi_2 \right) \cos(kz - \alpha).
 \end{aligned} \tag{12}$$

Substituting (7), (8) and (12) into boundary conditions (9), we obtain an infinite set of equations for the unknown coefficients. We shall obtain approximate solutions by using the method of moments. This method consists of two steps:

- Firstly, we truncate the expansions in (3) at some cut-off value N , i.e. we take into account

$$2 \times (4 + 6N) = 8 + 12N \tag{13}$$

unknowns and ignore the other ones. Except for $N = 0$ each order contains 12 unknowns (the expressions for $N = 0$ do not contain C_{j0} and F_{j0} , and therefore for $N = 0$ we only have 8 unknowns).

- Secondly, we solve the unknowns by means of the boundary conditions at the two interfaces. Using a linear theory, the boundary conditions (9) have to be evaluated at the interface S_1 given by $r_1 = a$ and S_2 given by $r_2 = a$. In these conditions, we have to evaluate all quantities at S_j in terms of ϕ_j : At the interface between thread 1 and the continuous phase, the quantities of the continuous phase should be expressed in a Fourier expansion of ϕ_1 only, whereas at the interface between thread 2 and the continuous phase it is the other way around. This implies that the product of a Bessel function and a Fourier expansion with respect to thread 2 should be expressed as product of Bessel functions and a Fourier expansion with respect to thread 1 or the other way around. To handle this, the following geometrical relations, derived from Fig. 3,

$$r_1 \cos \phi_1 = b - r_2 \cos \phi_2, \quad r_1 \sin \phi_1 = r_2 \sin \phi_2, \quad z_1 = z_2, \tag{14}$$

and the addition theorem of Bessel function [23] are used. The theorem states that for $r_1 < b$,

$$K_n(kr_2) \cos n\phi_2 = \sum_{m=-\infty}^{\infty} K_{n+m}(kb) I_m(kr_1) \cos m\phi_1. \quad (15)$$

This relation also holds with \cos replaced by \sin . Applying (14) and (15), the boundary conditions (9) at S_1 are now functions of r_1 and ϕ_1 containing terms with $\cos m\phi_1$ or $\sin m\phi_1$. Thus, the jump $\llbracket G \rrbracket$, with G representing one of the quantities $u, v, w, \sigma_{\phi r}, \sigma_{zr}, \sigma_{rr}$, is now described as

$$\llbracket G \rrbracket = \sum_{m=0}^{\infty} \llbracket G \rrbracket_m f_m(\phi_1), \quad (16)$$

where $f_m(\phi_1)$ is either $\cos m\phi_1$ or $\sin m\phi_1$, depending on whether G is odd (i.e., $v, \sigma_{\phi r}$) or even in ϕ_1 . In principle, boundary conditions (9) must be satisfied for every value of ϕ_1 , which results in an infinite number of equations. To make the problem tractable, we will not require pointwise satisfaction of the boundary conditions, but instead we require that the n th order moments, with $n \in [0, 1, \dots, N]$, of the boundary conditions are satisfied. Here, the n th order moment of $\llbracket G \rrbracket$ is defined as

$$\mathcal{M}_n\{\llbracket G \rrbracket\} = \int_0^{2\pi} \llbracket G \rrbracket f_n(\phi_1) d\phi_1. \quad (17)$$

By taking the first $N + 1$ moments of the boundary conditions (8 conditions for $N = 0$ and 12 conditions for every n with $1 \leq n \leq N$), we thus obtain $(8 + 12N)$ equations for $(8 + 12N)$ unknowns. Hence, in this way we obtain a finite set of equations for the unknown coefficients.

To illustrate and clarify the procedure, we work it out in the next section.

3. Solution for $N = 0$ and $N = 1$

In this section, we address the evaluation of the boundary conditions and the consequent matrix equation for the cases $N = 0$ and $N = 1$. From this, the generalization of the procedure to higher orders will become clear. Throughout this section, thread 1 will be taken as the frame of reference.

In the case $N = 0$ we have 8 unknowns. Then, we do not need the boundary conditions for v and $\sigma_{r\phi}$. From (3) we see that v_1 is identically zero. This is not so for v_c , as follows from (5). However, v_c is an odd function in ϕ_1 , and consequently the zero-moment is zero. The same holds for $\sigma_{r\phi}$.

The evaluation of boundary conditions at the interface S_1 requires that all quantities are expressed in terms of (r_1, ϕ_1) . As an example, we will work this out for the continuity of radial velocity. Requiring $\llbracket u \rrbracket = 0$, we find using (3)–(8):

$$\llbracket u \rrbracket = X(t) \cos kz + Y(r_2, \phi_1, \phi_2, t) \cos(kz - \alpha) = 0, \quad \text{where} \quad (18)$$

$$X(t) = \left(aI_0(ka) - \frac{2}{k}I_1(ka) \right) A_{10}(t) - I_1(ka)B_{10}(t) - \left(aK_0(ka) + \frac{2}{k}K_1(ka) \right) D_{10}(t) - K_1(ka)E_{10}(t) \quad \text{and} \quad (19)$$

$$Y(r_2, \phi_1, \phi_2, t) = \left[\left(r_2K_0(kr_2) + \frac{2}{k}K_1(kr_2) \right) D_{20}(t) + E_{20}(t)K_1(kr_2) \right] \cos(\phi_1 + \phi_2). \quad (20)$$

We note that X and Y are independent of z . Since (18) should hold for every value of z , there only exists a non-trivial solution of (18) if either $\alpha = 0$ or $\alpha = \pi$. Hence, we only meet with the two cases:

- (1) $\alpha = 0$, which is equivalent to $\varepsilon_{10}(t) = \varepsilon_{20}(t)$ in (1), implying that the threads will disintegrate in-phase;
- (2) $\alpha = \pi$, which is equivalent to $\varepsilon_{10}(t) = -\varepsilon_{20}(t)$ in (1), implying that the threads will disintegrate out-of-phase.

Condition (18) then reduces to

$$X(t) \pm Y(r_2, \phi_1, \phi_2, t) = 0, \quad (21)$$

where $+$ or $-$ sign corresponds to $\alpha = 0$ or $\alpha = \pi$, respectively. Here, Y is still written in terms of (r_2, ϕ_1, ϕ_2) , but by use of (14) and (15) we can evaluate r_2 and ϕ_2 as functions of ϕ_1 only. As an example, we show this for the first term of Y in (20):

$$\begin{aligned}
r_2 K_0(kr_2) \cos(\phi_1 + \phi_2) &= K_0(kr_2) [r_2 \cos \phi_1 \cos \phi_2 - r_2 \sin \phi_1 \sin \phi_2] \\
&= K_0(kr_2) [(b - r_1 \cos \phi_1) \cos \phi_1 - r_1 \sin \phi_1 \sin \phi_1] \\
&= \sum_{m=-\infty}^{\infty} K_m(kb) I_m(kr_1) \cos m \phi_1 [b \cos \phi_1 - r_1] \\
&= \sum_{m=-\infty}^{\infty} K_m(kb) I_m(kr_1) \left[\frac{b}{2} (\cos(m+1)\phi_1 + \cos(m-1)\phi_1) - r_1 \cos m \phi_1 \right]. \quad (22)
\end{aligned}$$

The calculation above is a correction on the calculation by Knops [21], who made the approximation $\cos(\phi_1 + \phi_2) \approx \cos \phi_1$. Along the same lines, we finally obtain that (21) can be written as

$$[\hat{u}] = [\hat{u}]_0 + \sum_{m=1}^{\infty} [\hat{u}]_m \cos m \phi_1 = 0. \quad (23)$$

Basically, we obtain an infinite set of equations namely $[\hat{u}]_m = 0$, for $m = 0, 1, 2, \dots$. Since we are in the first instance interested in the zero-order solution of (23), we approximate this expansion by taking the zero-moment of it, yielding $[\hat{u}]_0 = 0$, only. For instance, from expression (21) we find in this way at the interface S_1 :

$$\begin{aligned}
&A_{10} [ka I_0(ka) - 2I_1(ka)] - B_{10} k I_1(ka) - D_{10} [ka K_0(ka) + 2K_1(ka)] - E_{10} k K_1(ka) \\
&\pm D_{20} [kb K_1(kb) I_1(ka) - ka K_0(kb) I_0(ka) + 2K_0(kb) I_1(ka)] \pm E_{20} k K_0(kb) I_1(ka) = 0. \quad (24)
\end{aligned}$$

Evaluating the zero-moment of all boundary conditions and choosing an appropriate ordering of the unknowns, we arrive at the following matrix equation for the unknown coefficients:

$$\mathbf{MZ} = \begin{pmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \frac{\sigma}{2\mu_c k a^2} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}, \quad (25)$$

where the block matrices \mathbf{M}_{ij} have size 4 by 4, the vectors \mathbf{z}_1 and \mathbf{z}_2 are defined as

$$\mathbf{z}_i = (A_{i0}, B_{i0}, D_{i0}, E_{i0})^T \quad \text{for } i = 1, 2, \quad (26)$$

and the right-hand side vectors are given by

$$\mathbf{e}_i = (0, 0, 0, (1 - k^2 a^2) \varepsilon_{i0})^T \quad \text{for } i = 1, 2. \quad (27)$$

We remark here that \mathbf{z}_i and \mathbf{e}_i are time-dependent. To obtain the lower part of the matrix \mathbf{M} , we simply may replace coordinates referring to thread 1 by coordinates referring to thread 2. The expressions for the entries of the matrix are explicitly given in Appendix B.

Our calculations reveal that $\mathbf{M}_{12} = \mathbf{M}_{21}$, and, moreover, $\mathbf{M}_{22} = \mathbf{M}_{11}$. Thus, we can simplify the calculation as follows. For the case $\varepsilon_{10} = \varepsilon_{20}$, when the threads disintegrate in-phase, we find

$$\Delta^+ \mathbf{z}_1 = \frac{\sigma}{2\mu_c k a^2} \mathbf{e}_1 \quad \text{and} \quad \mathbf{z}_2 = \mathbf{z}_1, \quad (28)$$

with $\Delta^+ = \mathbf{M}_{11} + \mathbf{M}_{12}$, and for the case $\varepsilon_{10} = -\varepsilon_{20}$, when the threads disintegrate out-of-phase, we find

$$\Delta^- \mathbf{z}_1 = \frac{\sigma}{2\mu_c k a^2} \mathbf{e}_1 \quad \text{and} \quad \mathbf{z}_2 = -\mathbf{z}_1, \quad (29)$$

with $\Delta^- = \mathbf{M}_{11} - \mathbf{M}_{12}$. This solves the unknown coefficients of the zero-order solution. For instance, from (29) we have

$$A_{10}(t) = -\frac{\sigma}{2\mu_c k a^2} \frac{(1 - k^2 a^2) |\Delta_{41}^-|}{|\Delta^-|} \varepsilon_{10}(t), \quad (30)$$

where $|\cdot|$ denotes the determinant and Δ_{41}^- is the 3×3 submatrix of Δ^- which can be found by omitting the fourth row and the first column of Δ^- . The other expressions for these coefficients are given in Appendix C. We note that this coefficient is proportional to the amplitude $\varepsilon_{10}(t)$, and this evidently also holds for the other coefficients, as we are dealing with a linear problem. This observation will be important in the next discussion.

For the first-order approach with $N = 1$, we have twenty unknowns and six boundary conditions, two more than in the zero-order approach. The latter stem from the jump in velocity and shear stress in azimuthal direction. After application of transformations (14) and (15) and calculation of the zero and first moment, these six boundary conditions give rise to twenty

relations, i.e., 2×4 relations from the zero moment and 2×6 relations from the first moment. Similar as in the zero-order approach, we arrive at a matrix equation like the one in (25). The properties of the matrix \mathbf{M} are the same as in the previous case, except that the size of the block matrices \mathbf{M}_{ij} are now 10 by 10, the vectors \mathbf{z}_1 and \mathbf{z}_2 are defined as

$$\mathbf{z}_i = (A_{i0}, B_{i0}, D_{i0}, E_{i0}, A_{i1}, B_{i1}, C_{i1}, D_{i1}, E_{i1}, F_{i1})^T \quad \text{for } i = 1, 2,$$

and the right-hand side vectors are given by

$$\mathbf{e}_i = (0, 0, 0, (1 - k^2 a^2) \varepsilon_{i0}, 0, 0, 0, -k^2 a^2 \varepsilon_{i1}, 0, 0)^T \quad \text{for } i = 1, 2.$$

The same procedure as in (28) and (29) can be followed to solve the unknown coefficients. More details of the calculations for the first-order solution can be found in Gunawan et al. [24].

In the next section, the growth of the disturbance amplitude describing the (in)stability of the breaking up process of the threads is analyzed.

4. Stability analysis

In [1], as well as in [25], it is claimed that a cylindrical jet of water becomes unstable if the wavelength of the disturbances is greater than the circumference of the cylinder. Plateau therefore assumed that a cylindrical jet will break up into pieces of length equal to the circumference of the cylinder. This conclusion is incorrect, as Rayleigh showed [3]. Rayleigh found that the *mode of maximum instability* mainly determines the break-up of an unstable system. If a system is characterized by a number of unstable modes labelled by k_1, k_2, \dots, k_n , having amplitudes $e^{q_1 t}, e^{q_2 t}, \dots, e^{q_n t}$, $q_i = q(k_i)$, then the mode of maximum instability is the one pertinent to the q with the largest real part. In view of this, we will focus on finding the mode of maximum instability of the system under consideration. Again, thread 1 will be taken as the coordinate of reference.

In view of (1) and (3), we may write the disturbance amplitudes for every mode n as

$$\varepsilon_{1n}(t) = \varepsilon_{1n}(0) + \int_0^t u_{1n}(a, \tau) d\tau. \quad (31)$$

Since u_{10} is proportional to ε_{10} (see (7) and Appendix C), (31) reduces for $n = 0$ to

$$\varepsilon_{10}(t) = \varepsilon_{10}(0) + Q(b, \mu, k) \int_0^t \varepsilon_{10}(\tau) d\tau, \quad (32)$$

where Q is given by

$$Q(b, \mu, k) = \frac{1}{\varepsilon_{10}(0)} \left[\left(a I_0(ka) - \frac{2}{k} I_1(ka) \right) A_{10}(0) - I_1(ka) B_{10}(0) \right]. \quad (33)$$

The expressions for $A_{10}(0)$ and $B_{10}(0)$ can be found in Appendix C by taking $t = 0$. Taking the derivative of (32) with respect to t , we obtain the first-order scalar differential equation:

$$\frac{d}{dt} \varepsilon_{10} = Q(b, \mu, k) \varepsilon_{10}(t). \quad (34)$$

The behaviour of the solution of (34) depends on the the sign of $Q(b, \mu, k)$. This parameter acts as “the degree of instability” mentioned in [5]. We denote its dimensionless version by $q(b, \mu, k)$, defined as

$$q = \frac{2\mu_c a}{\sigma} Q(b, \mu, k), \quad (35)$$

where $\mu_c a / \sigma$ is the capillary time scale. For the case $\varepsilon_{10} = \varepsilon_{20}$ we use the notation q^+ , and for the case $\varepsilon_{10} = -\varepsilon_{20}$ the notation use q^- . They correspond with the in-phase and out-of-phase modes, respectively.

In general, the instability of the threads based on the solution up to and including the N th Fourier mode reduces to the matrix differential equation:

$$\frac{d\mathbf{E}}{dt} = \mathbf{Q}(b, \mu, k) \mathbf{E}, \quad (36)$$

where \mathbf{E} is the vector of the disturbance amplitudes and $\mathbf{Q}(b, \mu, k)$ is an $(N + 1)$ by $(N + 1)$ matrix depending on the distance b , the ratio of viscosities μ , and the wave number k . The instability of the threads is determined by the sign of the eigenvalue

of \mathbf{Q} with the largest real part. As before we denote these dimensionless eigenvalues for in-phase and out-of phase case by q^+ and q^- , respectively.

To get insight in the break up behaviour of the two threads system, we calculated the factor $q(b, \mu, k)$ for both $N = 0$ and $N = 1$. Fixing the geometrical parameter b and the material parameter μ , we determine the values q^+ and q^- by optimizing over k , since the (in)stability of the system is determined by the mode corresponding with the highest q^\pm -value. We denote this highest value of q^\pm by $q_{\max}^\pm = \max\{q^\pm(k; \mu, b)\}$. In Figs. 4–7 the values of q_{\max}^+ and q_{\max}^- , calculated this way, are given as

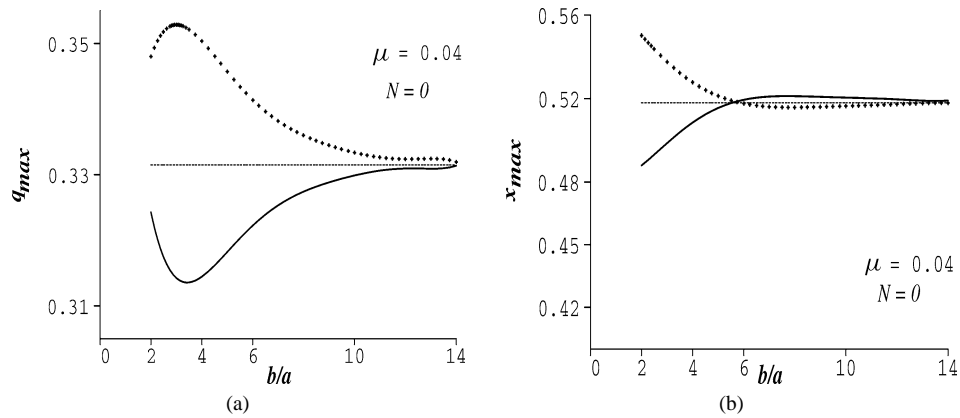


Fig. 4. Plots of q_{\max} and x_{\max} as functions of the relative distance b/a for $\mu = 0.04$ and $N = 0$. The solid and dotted lines correspond to in-phase and out-of-phase cases, respectively. (a) q_{\max} vs b/a ; (b) x_{\max} vs b/a .

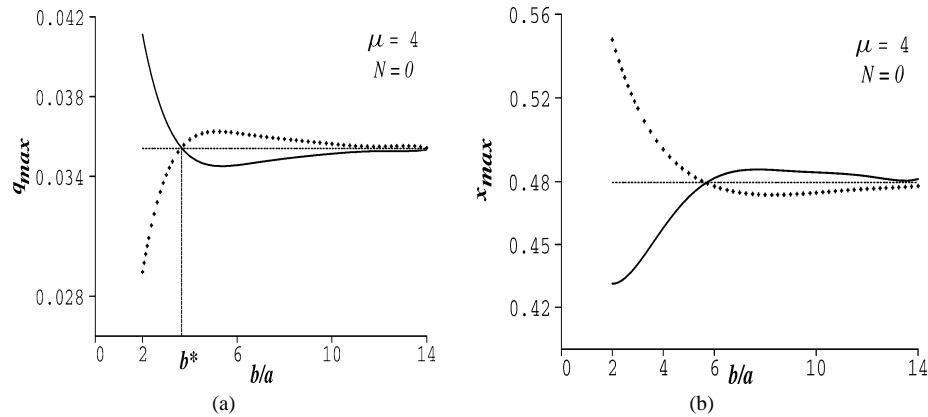


Fig. 5. Same information as in Fig. 4 but now for $\mu = 4$. (a) q_{\max} vs b/a ; (b) x_{\max} vs b/a .

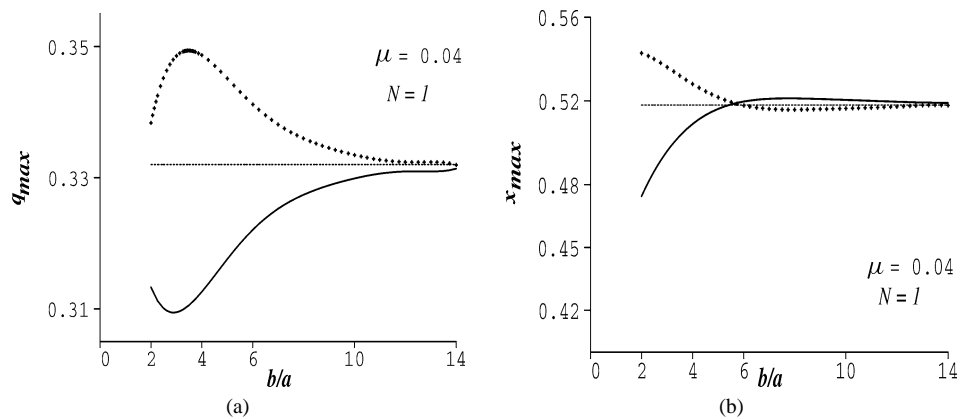


Fig. 6. Same information as in Fig. 4 but now for $N = 1$. (a) q_{\max} vs b/a ; (b) x_{\max} vs b/a .

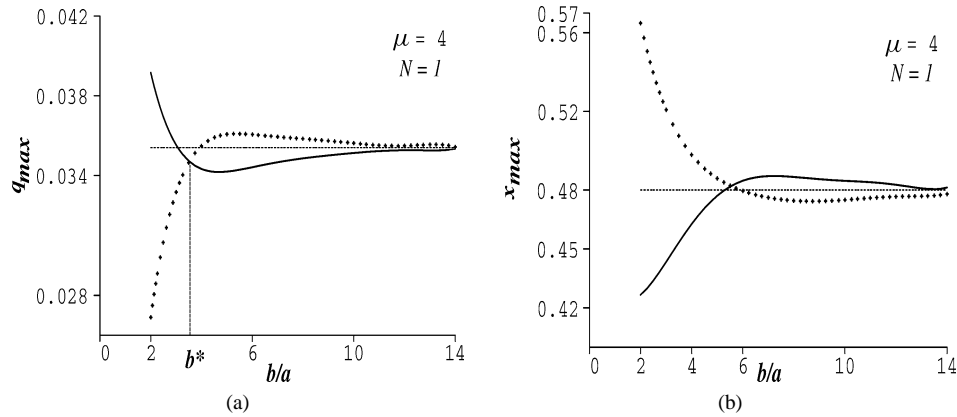


Fig. 7. Same information as in Fig. 4 but now for $\mu = 4$ and $N = 1$. (a) q_{\max} vs b/a ; (b) x_{\max} vs b/a .

functions of the relative distance $\hat{b} = b/a$ of the threads, for two values of μ . The case $\mu = 0.04$ corresponds to a situation in which the threads are less viscous than the surrounding fluid, whereas for $\mu = 4$ it is the other way around. The graph of x_{\max} , defined as the value of dimensionless wave number $x = ka$ corresponding to the highest q -value, is also presented in part (b) of those figures as a function of the relative distance \hat{b} .

For $\mu = 0.04$, q_{\max}^- is always greater than q_{\max}^+ . This implies that the threads will break up out-of-phase. However, for $\mu = 4$, the curves cross at a critical distance b^* . If $\hat{b} < b^*$ the threads will break up in-phase, whereas if $\hat{b} > b^*$ the threads will disintegrate out-of-phase. The discovery of the existence of this critical parameter b^* is a new result in this field. In the limiting case $\hat{b} \rightarrow \infty$, when the problem tends to the simple case of one thread, q_{\max}^+ and q_{\max}^- become equal to \hat{q}_{\max} , say. This indicates that in that case the threads will break up independently of each other, and then none of the phase differences are preferable. The limiting value \hat{q}_{\max} for the single thread case is denoted by the horizontal line in part (a) of the Figs. 4–7. Comparing the results in Figs. 4–7, we observe that the instability behaviour hardly differs for the zero and the first-order approach.

For the limiting case of one thread ($\hat{b} \rightarrow \infty$), we numerically calculated the maximum value x_{\max} of the wave number, which corresponds to the highest value of $q(k; \mu, \hat{b})$, for different values of the ratio of viscosity μ . Fixing the value of μ and taking the value of \hat{b} quite large, we determined x_{\max} and the results are plotted in Fig. 9 (dashed line). In this figure we also represent (solid line) Tomotika's result published in [5, Table V]. We see from this figure that the present results confirm the previous results of Tomotika for nearly all μ -values.

For the critical distance b^* , we present in Fig. 8 the graph as a function of the ratio of viscosities μ . If $b^* = 2$ the two threads touch each other. For the investigated range of μ values, we find that the higher is μ , the greater is b^* . The extension from $N = 0$ to $N = 1$ leads to very small corrections only.

The case $\mu = 1$ deserves special attention. In experiments, it is perfectly possible to find two fluids with identically equal viscosities that are immiscible. Stone et al. [9] considered this case analytically for the one thread system. In the present two thread system we find that for the zero-order approach the growth rates for the in-phase (q^+) and out-of phase break-up (q^-) are given by

$$q^{\pm}(b, 1, k) = \frac{\sigma}{\mu_c a} (1 - (ka)^2) \left\{ I_1(ka) K_1(ka) + \frac{ka}{2} [I_1(ka) K_0(ka) - I_0(ka) K_1(ka)] \right. \\ \left. \pm \left([ka I_0(ka) I_1(ka) - I_1^2(ka)] K_0(kb) - \frac{kb}{2} I_1^2(ka) K_1(kb) \right) \right\}. \quad (37)$$

In the limit $b \rightarrow \infty$, the two thread case tends to the single thread case and the growth rate is given by

$$\hat{q}(k) = \frac{\sigma}{\mu_c a} (1 - (ka)^2) \left\{ I_1(ka) K_1(ka) + \frac{ka}{2} [I_1(ka) K_0(ka) - I_0(ka) K_1(ka)] \right\}. \quad (38)$$

This growth rate is in perfect agreement with the result given by Stone et al. [9]. As follows from Fig. 8, the critical distance b^* of the two thread system with $\mu = 1$ is $b^* \approx 3.14$. We expect that this critical distance is exactly given by π and that this exact result is obtained if all orders of the Fourier expansion are taken into account. Similar suggestions can already be found in [2].

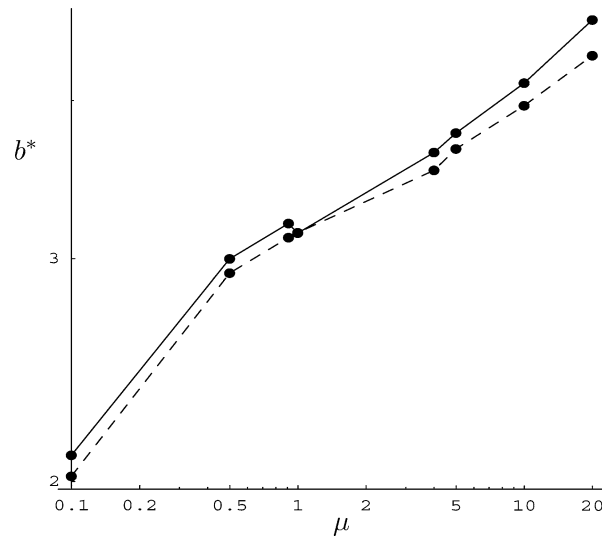


Fig. 8. Graph of the critical distance b^* as a function of the ratio of viscosities μ (on a logarithmic scale) for $N = 0$ (solid line) and $N = 1$ (dashed line).

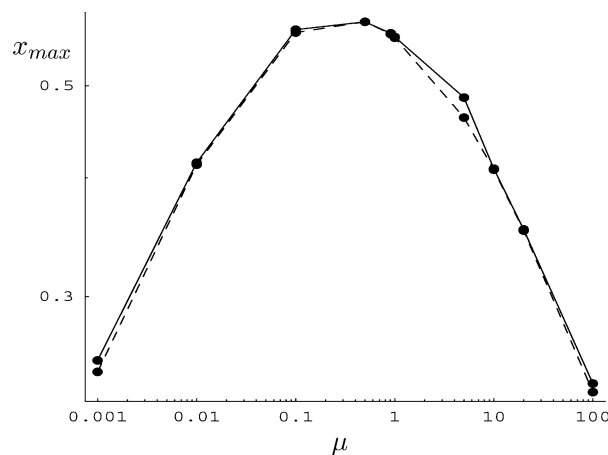


Fig. 9. Plot of x_{max} as a function of the ratio of viscosities μ (on a logarithmic scale) for the single thread case. The solid line represents the results from Tomotika [5, Table V], and the dashed line represents results based on present work in the zero-order approach ($N = 0$).

5. Conclusions

In this paper, we have described a general method to determine the degree of instability for a set of two parallel viscous threads immersed in a viscous liquid. The general solution of the underlying Stokes problem was decomposed into Fourier modes, and the boundary conditions then led to an infinite set of equations for the unknown coefficients. This set was truncated at finite N , and the truncated set was solved by use of the method of moments.

The method was evaluated for the zero-order ($N = 0$) and first-order ($N = 1$) solutions. As an illustration, the zero-order solution was worked out. The zero-order approximation leads to relatively simple equations for the unknown coefficients. No infinite series are involved in the entries of the matrix of coefficients as was found in Knops [21] due to redundant simplifications. The behaviour of the break-up process of the threads is characterized by the degree of instability q , which depends on the viscosity ratio μ , the wave number of the disturbances k and the distance between the two threads b . For large viscosity ratio, when the threads are more viscous than the matrix, we found a critical relative distance b^* . Below it the threads will break up in-phase, above it out-of-phase. The smaller the viscosity ratio, the more the out-of-phase break up is preferred. This is in accordance with experimental results reported in Knops [21,22]. However, the results in Knops et al. [22, Fig. 3] based on minimization of energy dissipation arguments disagree with the present findings. Not only the value of the critical b^* is

different, but also the quantitative behaviour, since we find that out-of-phase break-up takes place for $b > b^*$, whereas in [22] this is found for $b < b^*$.

We have also performed the instability analysis for $N = 1$ which led to a large amount of arithmetical work (it is recommended to look at the report of Gunawan et al. [24]). The degree of instability is determined by the largest real part of the eigenvalue of a matrix which entries depend on three parameters above. We found that the extension from $N = 0$ to $N = 1$ leads only to minor quantitative corrections in the results.

From the point of view of blend production the present results may provide important insights for control of the production process. Two aspects deserve mentioning: characteristic drop formation times and spatial distributions of the droplets. As for formation time, the time scale of the dynamical process is governed by the value of the q -factor, plotted in Figs. 4–7. In most extrusion devices the blend is only a restricted time in the molten state. As soon as it is cooled down, the spatial droplet distribution at that moment will freeze in. So, from industry there is a need for fast formation processes. From the results summarized in those figures it is clear that high values for μ and small values for b are highly favourable for speeding up the break-up process.

As for the spatial distribution of the droplets, it is highly important to know that either in-phase or out-of-phase break-up may occur. If the present results also apply to system with many threads, it must be possible to produce blends with the droplets either on a cubic grid, resulting from in-phase break-up, or on a body centered cubic grid, resulting from out-of-phase break-up, just by controlling the two parameters μ and b in an appropriate way. So, this implies in practice that important properties of the blends can be adjusted quite easily, since both μ and b , which is essentially determined by the volume fraction of the big drops in the initial situation, are simply controlled in production conditions. These considerations are based on our expectation that the present results for two threads may be transferred to many-threads systems. This is subject of ongoing research, in which droplet formation is studied for systems with a layer of equally spaced threads. The same analysis will also be applied to three-dimensional arrays of threads.

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Appendix A. Some expressions related to the addition theorem for Bessel functions

Below, some relations for Bessel functions used in the evaluation of the boundary conditions are given. The prime denotes derivation with respect to x where $x = kr_1$, $[-]_{r_1}$ indicates derivation with respect to r_1 , $y = kb$, and $z = kr_2$.

$$\begin{aligned}
 K_1(z) \cos(\phi_1 + \phi_2) &= \sum_{m=-\infty}^{\infty} K_{m+1}(y) I_m(x) \cos(m+1)\phi_1, \\
 r_2 K_1(z) &= \sum_{m=-\infty}^{\infty} K_{m+1}(y) I_m(x) [b \cos m\phi_1 - r_1 \cos(m+1)\phi_1], \\
 K_0(z) &= \sum_{m=-\infty}^{\infty} K_m(y) I_m(x) \cos m\phi_1, \quad [K_0(z)]_{r_1} = \sum_{m=-\infty}^{\infty} k K_m(y) I'_m(x) \cos m\phi_1, \\
 [K_1(z) \cos(\phi_1 + \phi_2)]_{r_1} &= \sum_{m=-\infty}^{\infty} k K_{m+1}(y) I'_m(x) \cos(m+1)\phi_1, \\
 [r_2 K_1(z)]_{r_1} &= \sum_{m=-\infty}^{\infty} k K_{m+1}(y) I'_m(x) [b \cos m\phi_1 - r_1 \cos(m+1)\phi_1] - \sum_{m=-\infty}^{\infty} K_{m+1}(y) I_m(x) \cos(m+1)\phi_1, \\
 [r_2 K_0(z) \cos(\phi_1 + \phi_2)]_{r_1} &= \sum_{m=-\infty}^{\infty} k K_m(y) I'_m(x) \left[b \left(\frac{\cos(m+1)\phi_1 + \cos(m-1)\phi_1}{2} \right) - r_1 \cos m\phi_1 \right] \\
 &\quad - \sum_{m=-\infty}^{\infty} K_m(y) I_m(x) \cos m\phi_1.
 \end{aligned}$$

Appendix B. Explicit expressions for the block matrices \mathbf{M}_{11} and \mathbf{M}_{12} in (25)

Here the following notations are used: $x = ka$, $y = kb$, $\mu = \mu_d/\mu_c$, $I'_1(x) = I_0(x) - I_1(x)/x$ and $K'_1(x) = -K_0(x) - K_1(x)/x$.

$$\mathbf{M}_{11} = \begin{pmatrix} xI_0(x) - 2I_1(x) & -kI_1(x) & -(xK_0(x) + 2K_1(x)) & -kK_1(x) \\ -xI_1(x) & kI_0(x) & -xK_1(x) & -kK_0(x) \\ -2\mu xI'_1(x) & 2\mu kI_1(x) & -2xK'_1(x) & 2kK_1(x) \\ \mu(xI_1(x) - 2I'_1(x))/k & -\mu I'_1(x) & (xK_1(x) - 2K'_1(x))/k & -K'_1(x) \end{pmatrix},$$

$$\mathbf{M}_{12} = \begin{pmatrix} 0 & 0 & yK_1(y)I_1(x) - xK_0(y)I_0(x) + 2K_0(y)I_1(x) & kK_0(y)I_1(x) \\ 0 & 0 & -yK_1(y)I_0(x) + xK_0(y)I_1(x) & -kK_0(y)I_0(x) \\ 0 & 0 & -2yK_1(y)I_1(x) + 2xK_0(y)I_0(x) - 2K_0(y)I_1(x) & -2kK_0(y)I_1(x) \\ 0 & 0 & (yK_1(y)I'_1(x) - xK_0(y)I_1(x) + 2K_0(y)I'_1(x))/k & K_0(y)I'_1(x) \end{pmatrix}.$$

Appendix C. Explicit solution of (28) and (29)

We denote by Δ_{mn}^s the 3×3 submatrix of Δ^s obtained by omitting the m th row and the n th column of Δ^s ; $|\cdot|$ indicates the determinant of a matrix. The label s can be either $+$ or $-$.

$$A_{10}(t) = -\frac{\sigma}{2\mu_c x a} \frac{(1-x^2)|\Delta_{41}^s|}{|\Delta^s|} \varepsilon_{10}(t), \quad B_{10}(t) = \frac{\sigma}{2\mu_c x a} \frac{(1-x^2)|\Delta_{42}^s|}{|\Delta^s|} \varepsilon_{10}(t),$$

$$D_{10}(t) = -\frac{\sigma}{2\mu_c x a} \frac{(1-x^2)|\Delta_{43}^s|}{|\Delta^s|} \varepsilon_{10}(t), \quad E_{10}(t) = \frac{\sigma}{2\mu_c x a} \frac{(1-x^2)|\Delta_{44}^s|}{|\Delta^s|} \varepsilon_{10}(t).$$

References

- [1] J. Plateau, Acad. Sci. Bruxelles Mém. 23 (1849).
- [2] M. Plateau, Statique des Liquides, Gauthier-Villars, Paris, 1873.
- [3] L. Rayleigh, On the stability of liquid jets, Proc. London Math. Soc. 10 (1878) 4–18.
- [4] L. Rayleigh, On the instability of a cylinder of viscous liquid under the capillary force, Phil. Mag. 34 (1892) 145–154.
- [5] S. Tomotika, On the instability of a cylindrical thread of a viscous liquid surrounded by another viscous fluid, Proc. Roy. Soc. A 150 (1935) 322–337.
- [6] S. Tomotika, Breaking up of a drop of viscous liquid immersed in another viscous fluid which is extending at a uniform rate, Proc. Roy. Soc. A 153 (1936) 302–318.
- [7] B.J. Meister, G.F. Scheele, Generalized solution of the Tomotika stability analysis for a cylindrical jet, AIChE J. 13 (1967) 682–688.
- [8] C.M. Kinoshita, H. Teng, S.M. Masutani, A study of the instability of liquid jets and comparison with Tomotika's analysis, Int. J. Multiphase Flow 20 (1994) 523–533.
- [9] H.A. Stone, M.P. Brenner, Note on the capillary thread instability for fluids of equal viscosities, J. Fluid Mech. 318 (1996) 373–374.
- [10] J. Eggers, Nonlinear dynamics and breakup of free surface flows, Rev. Mod. Phys. 69 (1997) 865–929.
- [11] K.O. Mikaelian, Effect of viscosity on Rayleigh–Taylor and Richtmyer–Meshkov instabilities, Phys. Rev. E 47 (1993) 375–383.
- [12] K.O. Mikaelian, Rayleigh–Taylor instability in finite-thickness fluids with viscosity and surface tension, Phys. Rev. E 54 (1996) 3676–3680.
- [13] D.T. Papageorgiou, On the breakup of viscous liquid threads, Phys. Fluids 7 (1995) 1529–1544.
- [14] J. Eggers, Universal pinching of 3D axisymmetric free-surface flow, Phys. Rev. Lett. 71 (1993) 3458–3460.
- [15] J.R. Lister, H.A. Stone, Capillary breakup of a viscous thread surrounded by another viscous fluid, Phys. Fluids 10 (1998) 2758–2764.
- [16] J. Eggers, Theory of drop formation, Phys. Fluids 7 (1995) 941–953.
- [17] Y. Navot, Critical behavior of drop breakup in axisymmetric viscous flow, Phys. Fluids 11 (1999) 990–996.
- [18] S. Guido, M. Minale, P.L. Maffettone, 3D shape of a Newtonian drop in a Newtonian liquid under transient shear flows, in: Proc. XIIIth International Congress on Rheology, Vol. 2, Cambridge Univ. Press, Cambridge, UK, 2000, pp. 42–44.
- [19] Y. Renardy, J. Li, Breakup under shear of a viscous drop in Bingham liquid, in: Proc. XIIIth International Congress on Rheology, Vol. 2, Cambridge Univ. Press, Cambridge, UK, 2000, p. 336.
- [20] S. Wannaborworn, M.R. Mackley, Deformation and breakup of viscous drops in oscillatory shear, in: Proc. XIIIth International Congress on Rheology, Vol. 2, Cambridge Univ. Press, Cambridge, UK, 2000, pp. 250–252.

- [21] Y.M.M. Knops, Morphology Development in Polymer Blends: The Hydrodynamic Interaction Between Disintegrating Threads, Final Report of the Postgraduate Programme Mathematics for Industry, SAI, Eindhoven University of Technology, Eindhoven, 1997, ISBN 90-5282-759-1.
- [22] Y.M.M. Knops, J.J.M. Slot, P.H.M. Elemans, M.J.H. Bulters, Simultaneous breakup of multiple viscous threads surrounded by viscous liquid, *AIChE J.* (2001), in print.
- [23] C.J. Tranter, Bessel Functions with Some Physical Applications, English Univ. Press, London, 1968.
- [24] A.Y. Gunawan, J. Molenaar, A.A.F. van de Ven, Dynamics of a Polymer Blend Driven by Surface Tension: Part 3: The First Order Solution, RANA 01-21, Eindhoven University of Technology, 2001, ISSN 0926-4507.
- [25] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, Clarendon Press, Oxford, 1961, Chapter XII.